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# THE STABILITY OF THE STEADY-STATE MOTIONS OF A SYSTEM WITH PSEUDOCYCLICAL COORDINATES*

V.A. ATANASOV and L.K. LILOV

The sufficient conditions for the asymptotic stability of the steady-state motions of a mechanical system with pseudocyclical coordinates, by means of forces acting on these coordinates when dissipation with respect to the positional coordinates is present, are formulated. Both gyroscopically connected and unconnected systems are considered. The results are used to study the possible stabilization of the steady-state motion of an unbalanced rotor on a flexible shaft.

1. Consider a holonomic scleronomic mechanical system with $n$ degrees of freedom. Let $q_j$ be the generalized coordinates of the system, $q_j^{\cdot}$, $p_j^{\cdot}$ the generalized velocities and momenta $(j = 1, \ldots, n)$, $T$ and $\pi$ the kinetic and potential energies respectively, and $L = T - \pi$ the Lagrange function. Let non-potential forces $Q_j$ $(j = 1, \ldots, n)$ as well as potential forces, act on the system. It will be assumed throughout that there are coordinates $q_\alpha$ (always, $\alpha = m + 1, \ldots, n; m < n$) which do not appear explicitly in the expression for the Lagrange function $L$ $(\partial L/\partial q_\alpha = 0)$. We also assume that the forces acting on the system are likewise independent of these coordinates, which we shall call pseudocyclical. The remaining coordinates $q_i$ $(i = 1, \ldots, m)$ are positional. The generalized non-potential forces $Q_i$ $(i = 1, \ldots, m)$ will be regarded as dissipative with respect to the generalized velocities; the dissipation may be incomplete, or, in particular, may be zero.

When there are no forces $Q_\alpha$, acting on the pseudocyclical coordinates, the system can perform a steady-state motion, in which the potential coordinates $q_i$ and the pseudocyclical velocities $q_\alpha^{\cdot}$ remain constant, while the pseudocyclical coordinates $q_\alpha$ vary linearly with time. Our main problem is to find the conditions under which the steady-state motion can be stabilized up to asymptotic stability with respect to the positional coordinates and all the velocities, by means of forces $Q_\alpha$ which act only on the pseudocyclical coordinates.

This problem was first considered in /1, 2/ when studying mechanical systems when there is no dissipation. It was proposed in /3/ to choose the forces $Q_\alpha$ in such a way that a pre-assigned linear manifold proved to be an invariant asymptotically stable integral manifold for the system of linearized differential equations of the perturbed motion. If the linearized system is then asymptotically stable on the manifold with respect to the positional coordinates, these forces $Q_\alpha$ then solve the problem of the asymptotic stability of the steady-state motion. This method of constructing the stabilizing signals was used to study the stability of any steady-state motions of gyroscopically unconnected systems /3/ and the trivial steady-state motions of gyroscopically connected systems /4/. Different methods may be used to conctruct the stabilizing signals, in particular the method given in /5/.

However, before trying to construct the stabilizing signals, we must ask the fundamental questions as to whether a given steady-state motion can in fact be stabilized by forces which act on the pseudocyclical coordinates. Below, we state sufficient conditions for this problem to be solvable for any systems with pseudocyclical coordinates when there are dissipative forces on the positional coordinates.

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2. We choose the Rouse variables

$$q = (q_1, \ldots, q_m)^T, \quad q_* = (q_{m+1}, \ldots, q_n)^T$$
$$q^{\cdot} = (q_1^{\cdot}, \ldots, q_m^{\cdot})^T, \quad p = (p_{m+1}, \ldots, p_n)^T$$

On substituting into the expression for the kinetic energy

$$T = \frac{1}{2} \| q^{\cdot T}, \quad q_*^{\cdot T} \| \begin{Vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{Vmatrix} \begin{Vmatrix} q^{\cdot} \\ q_*^{\cdot} \end{Vmatrix}$$

the dependence

$$q_*^{\cdot} = A_{22}^{-1} (p - A_{21} q^{\cdot}) \tag{2.1}$$

we obtain

$$T = {}^1/_2 q^{\cdot T} (A_{11} - A_{12} A_{22}^{-1} A_{21}) q^{\cdot} + {}^1/_2 p^T A_{22}^{-1} p \tag{2.2}$$

Here, $A_{11} = A_{11}^T, A_{21} = A_{12}^T, A_{22} = A_{22}^T$ are submatrices of the positive definite $(n \times n)$ matrix of the kinetic energy, of dimensions $m \times m, r \times m, r \times r \; (r = n - m)$ respectively. Using (2.1) and (2.2), we can write the Rouse function as

$$R = R(q, q^{\cdot}, p) = T - \pi + p^T q_*^{\cdot} = R_2 + R_1 - W$$
$$R_2 = {}^1/_2 q^{\cdot T} A q^{\cdot}, \; A = A_{11} - A_{12} A_{22}^{-1} A_{21}$$
$$R_1 = p^T A_{22}^{-1} q^{\cdot} = g^T q^{\cdot}, \; g = A_{12} A_{22}^{-1} p = \| g_1, \ldots, g_m \|^T$$
$$W = {}^1/_2 p^T A_{22}^{-1} p + \pi(q)$$

Study of the motion of the mechanical system amounts to studying the system of equations

$$\begin{aligned} &\frac{d}{dt} \frac{\partial R_2}{\partial q_i^{\cdot}} - \frac{\partial R_2}{\partial q_i} = \frac{\partial W}{\partial q_i} + \sum_{s=1}^{n} g_{is} q_s^{\cdot} - \sum_{\alpha=m+1}^{n} \frac{\partial g_i}{\partial p_\alpha} Q_\alpha + Q_i \\ &dp_\alpha / dt = Q_\alpha \\ &(g_{is} = \partial g_s / \partial q_i - \partial g_i / \partial q_s = -g_{si}; \quad i, s = 1, \ldots, m) \end{aligned} \tag{2.3}$$

which, on integration, gives the $q_\alpha$ by means of the quadratures

$$q_\alpha = \int (\partial R / \partial p_\alpha) \, dt + c_\alpha'$$

When $Q_\alpha = 0$, the first $m$ equations of (2.3) can be regarded as the equations of a fictitious mechanical system, whose kinetic and potential energies are the functions $R_2$ and $W$ respectively, and on which act additionally gyroscopic and dissipative forces. We call this the reduced system; it can have equilibrium positions which correspond to the steady-state motions of the initial system, when the positional coordinates and the pseudocyclical momenta remain constant, while the pseudocyclical coordinates vary linearly with time. The possible steady-state motions of the system are given by the conditions

$$\partial W / \partial q_i \,|_{q=q^\circ, \, p=c} = 0 \quad (i = 1, \ldots, m) \tag{2.4}$$

while the control must satisfy the equations

$$Q_\alpha \,|_{q=q^\circ, \, p=c} = 0$$

We linearize Eqs.(2.3) in the neighbourhood of the steady-state motion. Putting $q = q^\circ + \xi, p = c + \eta, (Q_{m+1}, \ldots, Q_n)^T = (u_{m+1}, \ldots, u_n)^T$, we obtain

$$\begin{aligned} &A\xi^{\cdot\cdot} + (D - G)\xi^{\cdot} + C\xi + N\eta + \Gamma u = 0, \quad \eta^{\cdot} = u \\ &A = A(q^\circ), \; C = \| \partial^2 W (q_0, c) / \partial q_i \partial q_s \|, \quad G = \| g_{is} (q^\circ, c) \| = -G^T \\ &N = \| \partial^2 W (q^\circ, c) / \partial q_i \partial p_\alpha \|, \quad \Gamma = A_{12} (q^\circ) A_{22}^{-1} (q^\circ)' \\ &(i, s = 1, \ldots, m) \end{aligned} \tag{2.5}$$

where $D$ is the $(m \times m)$ matrix of dissipative forces $Q_i \; (i = 1, \ldots, m)$.

Let $\lambda_i$ be the roots of the equation

$$\det \| C - \lambda A \| = 0 \tag{2.6}$$

We can always make a change of variables $z = \Phi\xi$ in such a way that system (2.5) takes the form /6/

$$z^{\cdot\cdot} + (D_1 - G_1) z^{\cdot} + \Lambda z + N_1 \eta + \Gamma_1 u = 0, \quad \eta^{\cdot} = u \tag{2.7}$$

$$D_1 = \Phi^T D\Phi, \quad G_1 = \Phi^T G\Phi, \quad N_1 = \Phi^T N, \quad \Gamma_1 = \Phi^T\Gamma$$
$$\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_m)$$

We write system (2.7) in the normal form

$$v^{\cdot} = Lv + Ku, \quad v = (z^T, z^{\cdot T}, \eta^T)^T$$
$$L = \left\| \begin{array}{cc|c} 0 & E_m & 0 \\ -\Lambda & G_1 - D_1 & -N_1 \\ \hline 0 & 0 & 0 \end{array} \right\|, \quad K = \left\| \begin{array}{c} 0 \\ -\Gamma_1 \\ \hline E_r \end{array} \right\|$$

Here and throughout, $E_k$ is the unit $(k \times k)$ matrix. The condition

$$\mathrm{rank}\,(K, LK, \ldots, L^{2m+r-1}K) = 2m + r \tag{2.8}$$

for the pair $(L, K)$ to be completely controllable, is obviously sufficient for the asymptotic stability of the steady-state motion $q = q^\circ, p = c$ with respect to the positional coordinates and all the velocities. For a linear system (2.5) with no dissipation $(D = 0)$, condition (2.8) is necessary, since the characteristic equation $\det \| L - \mu E_{2m+r} \| = 0$ of the system contains only even powers of $\mu$ due to the relation $G^T = -G$, and hence, when condition (2.8) is violated, the uncontrolled part may not be asymptotically stable.

If we put

$$P = \left\| \begin{array}{cc} 0 & E_m \\ -\Lambda & G_1 - D_1 \end{array} \right\|, \quad Q = \left\| \begin{array}{c} 0 \\ -N_1 \end{array} \right\|, \quad B = \left\| \begin{array}{c} 0 \\ -\Gamma_1 \end{array} \right\|$$

it can be shown that

$$L^i K = \left\| \begin{array}{c} P^{i-1}(PB + Q) \\ 0 \end{array} \right\| \quad (i = 1, \ldots, 2m + r - 1)$$

so that (2.8), recalling that $r = n - m \geqslant 1$, is equivalent to the condition

$$\mathrm{rank}\, S = 2m, \quad S = \| PB + Q, \ P(PB + Q), \ldots, P^{2m-1}(PB + Q) \|$$

On now using the expression

$$PB + Q = -R \| \Gamma_1^T, N_1^T \|^T, \quad R = \left\| \begin{array}{cc} E_m & 0 \\ G_1 - D_1 & E_m \end{array} \right\|$$

and the obvious relation $\mathrm{rank}\, S = \mathrm{rank}\,(R^{-1}S)$, we can reduce (2.8) to the equivalent condition

$$\mathrm{rank}\, \| B_1, P_1 B_1, \ldots, P_1^{2m-1} B_1 \| = 2m \tag{2.9}$$
$$P_1 = R^{-1}PR = \left\| \begin{array}{cc} G_1 - D_1 & E_m \\ -\Lambda & 0 \end{array} \right\|, \quad B_1 = \left\| \begin{array}{c} -\Gamma_1 \\ -N_1 \end{array} \right\|$$

We have thus proved the following theorem.

*Theorem.* Complete controllability of the pair $(P_1, B_1)$ is a sufficient condition for the asymptotic stability of the steady-state motion of a mechanical system with pseudocyclical coordinates with respect to the positional coordinates and all the velocities, by means of forces which act only on the pseudocyclical coordinates, when the positional coordinates are subject to dissipative forces.

3. Consider the case of a gyroscopically unconnected system, i.e., $A_{12} = 0$, so that $G_1 = 0, \Gamma_1 = 0$. System (2.7) now has the form

$$z^{\cdot\cdot} + D_1 z^{\cdot} + \Lambda z + N_1 \eta = 0, \quad \eta^{\cdot} = u \tag{3.1}$$

By our theorem, the zero solution $z = z^{\cdot} = 0, \ \eta = 0$ can be stabilized up to asymptotic stability with respect to $z$ and $z^{\cdot}$ is the pair $(P_1, B_1)$, where

$$P_1 = \left\| \begin{array}{cc} -D_1 & E_m \\ -\Lambda & 0 \end{array} \right\|, \quad B_1 = \left\| \begin{array}{c} 0 \\ -N_1 \end{array} \right\|$$

is completely controllable.

Consider the corresponding system of differential equations

$$y^{\cdot} = P_1 y + B_1 w \quad (3.2)$$

in which we make the change of variables

$$y = R^{-1} z, \quad R = \left\| \begin{matrix} E_m & 0 \\ -D_1 & E_m \end{matrix} \right\|$$

System (3.2) clearly transforms into a system which is precisely the first matrix equation of system (3.1), written in the normal form, if we put $\eta = w$ in it. We thus arrive at the following corollary.

*Corollary.* The condition for complete controllability of the system

$$z^{\cdot\cdot} + D_1 z^{\cdot} + \Lambda z = -N_1 \eta$$

in which the perturbations $\eta$ of the cyclical momenta and regarded as the control, is a sufficient condition for asymptotic stability of the steady-state motion of a gyroscopically unconnected system with respect to the positional coordinates and all the velocities, by means of forces which act on the pseudocyclical coordinates when the positional coordinates are subject to dissipative forces.

There are no control terms in (3.1) in the case of the trivial steady-state motion when Eqs. (2.5) on the hyperplane $q = q^{\circ}$ have the form /3/

$$\partial\pi/\partial q_i = 0, \quad \partial \| A_{22}^{-1} \| / \partial q_i = 0 \quad (i = 1, \ldots, m)$$

In this case, $N_1 = 0$, whence it follows that any steady-state motion that is unstable to a first approximation cannot be stabilized by linear forces which act on the pseudocyclical coordinates. If all the roots $\lambda_i > 0$ $(i = 1, \ldots, m)$, then, by one of the Kelvin-Chetayev theorems, we can always arrange for asymptotic stability of the zero solution by forces of total dissipation $(D_1 > 0)$. Completeness of the dissipation is not, however, a necessary condition, and we can arrange for asymptotic stability by suitably chosen forces of partial dissipation with a degenerate matrix $D_1 \geqslant 0$, whose rank is equal to the maximum multiplicity of the eigenvalues $\lambda_i$ /7, 8/. In particular, if $\lambda_1 = \lambda_2 = \ldots = \lambda_m$, the matrix $D_1$ is as degenerate as possible and its rank is unity.

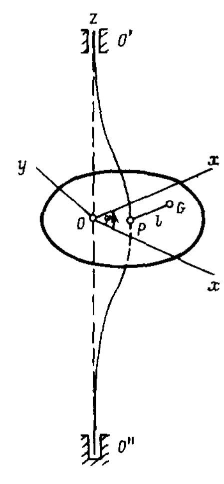

Fig.1

4. As an example, consider the possibility of asymptotic stabilization by the turning moment (see Fig.1) of the rotation of a rotor which is clamped eccentrically to a shaft. As in /9/, we shall assume that the rotor performs plane-parallel motion, and we introduce the coordinate system $Oxy$ into the plane of motion; the origin $O$ is the point of intersection of the plane with the straight line connecting the shaft bearings, $O'O''$ while the $x$ axis is parallel to the segment $PG$, where $G$ is the centre of mass of the rotor, and $P$ is the point at which it is clamped to the shaft. Assuming that the angular velocity of the rotor rotation does not exceed a certain value, a turning moment was found in /9/ which asymptotically stabilizes the steady-state motion of the rotor, in which the rotor rotates with constant angular velocity, while its centre of mass remains fixed in the uniformly rotating $Oxy$ system of coordinates.

We shall show that this steady-state motion can be asymptotically stabilized for all values of the angular velocity except one.

The kinetic and potential energies of the system are

$$T = {}^1/_2 m\, [x^{\cdot 2} + y^{\cdot 2} + 2\varphi^{\cdot} (xy^{\cdot} - yx^{\cdot}) + \varphi^{\cdot 2} (x^2 + y^2)] + {}^1/_2 J \varphi^{\cdot 2}$$
$$\pi = {}^1/_2 c\, [(x + l)^2 + y^2]$$

where $m$ is the mass and $J$ is the central moment of inertia of the rotor, $x$ and $y$ are the coordinates of its centre of mass $G$ in the $Oxy$ system, $\varphi$ is the angle between the $x$ axis and the fixed axis $X$ in the plane of motion of the point $G$, $l$ is the length of segment $PG$, and $c$ is the coefficient of elasticity of the shaft. We assume that the system is acted on by the internal resistance force $(-ax^{\cdot}, -ay^{\cdot})$, applied to a point of the shaft ($a$ is the coefficient of internal resistance), and by the controlling moment $M(x, x^{\cdot}, y, y^{\cdot}, \varphi^{\cdot})$, which has to be found.

The matrices in (2.2) are

$$A_{11} = \left\| \begin{matrix} m & 0 \\ 0 & m \end{matrix} \right\|, \quad A_{12} = \left\| \begin{matrix} -my \\ mx \end{matrix} \right\|, \quad A_{22}^{-1} = \frac{1}{J + m(x^2 + y^2)}$$

$$q = (x, y)^T, \quad p = \partial T/\partial\varphi^{\cdot} = \varphi^{\cdot} [J + m(x^2 + y^2)] + m(xy^{\cdot} - yx^{\cdot})$$

The steady-state motions of the system are

$$q = q^\circ = (x_0, y_0)^T, \quad p = k_0 = \omega_0 [J + m(x_0^2 + y_0^2)]$$

where $\omega_0$ is the angular velocity of rotor rotation, and are found from system (2.4), which in the present problem has the form

$$\varkappa x_0 + cl = 0, \quad \varkappa y_0 = 0; \quad \varkappa = c - m\omega_0^2 \tag{4.1}$$

while the control must satisfy the condition

$$M(x_0, 0, y_0, 0, \omega_0) = 0 \tag{4.2}$$

After this, the problem of choosing the control amounts to finding the feedback coefficients in the linear part of the dependence of the control moment $M$ on the disturbances of the positional coordinates and all the velocities.

In the present case of an unbalanced rotor $(l \neq 0)$, system (4.1) is only meaningful under the condition $\omega_0^2 \neq c/m$, when $x_0 = -cl/\varkappa$, $y_0 = 0$, while $\omega_0$ is given by condition (4.2). The matrices in Eqs.(2.5) are

$$A = \begin{Vmatrix} m & 0 \\ 0 & J \end{Vmatrix}, \quad C = \begin{Vmatrix} \varkappa + 4\nu m\omega_0^2 x_0^2 & 0 \\ 0 & \varkappa \end{Vmatrix}$$

$$G = \begin{Vmatrix} 0 & 2\nu\omega_0 J \\ -2\nu\omega_0 J & 0 \end{Vmatrix}, \quad D = \begin{Vmatrix} a & 0 \\ 0 & a \end{Vmatrix}$$

$$N = \begin{Vmatrix} -2\nu\omega_0 x_0 \\ 0 \end{Vmatrix}, \quad \Gamma = \begin{Vmatrix} 0 \\ -\nu x_0^2 \end{Vmatrix}; \quad \nu = \frac{m}{J + m x_0^2}$$

The roots of the secular Eq.(2.6) are

$$\lambda_1 = \varkappa/m + 4\nu m\omega_0^2 x_0^2, \quad \lambda_2 = \varkappa/(\nu J)$$

When $\varkappa > 0$, i.e., under the condition

$$\omega_0 < \sqrt{c/m} \tag{4.3}$$

the roots $\lambda_1, \lambda_2$ are both positive. Under this condition, the control $u = k\eta$ $(k < 0)$ makes the manifold $\eta = 0$ an asymptotically stable invariant manifold of system (2.5) and thereby asymptotically stabilizes the steady-state motion /9/.

Condition (4.3) is not necessary for asymptotic stabilization.

For, make the transformation

$$z = \Phi\xi, \quad \Phi = \begin{Vmatrix} m^{-1/2} & 0 \\ 0 & (\nu J)^{-1/2} \end{Vmatrix}$$

Condition (2.9) now reduces to the condition

$$(c - m\omega_0^2) + 4\omega_0^2 a = 0 \tag{4.4}$$

which, with the exception of the case $\omega_0 = \sqrt{c/(m - 4a)}$, is always satisfied.

In short, given any steady-state angular velocity $\omega_0$, except for the value mentioned, we can find a control moment which asymptotically stabilizes the motion. It must be said that in this case the internal friction forces of the unstabilized rotor do not have a significant influence on the asymptotic stabilization, since condition (4.4) also holds when $a = 0$.

Now assume that the rotor is stabilized, i.e., the centre of mass is the same as the geometric centre $(l = 0)$. System (4.1) has the solution $\omega_0 = \sqrt{c/m}$, $y_0 = 0$, while $x_0$ is found in such a way that condition (4.2) holds. If there is no internal friction $(a = 0)$, condition (4.4) may not hold, so that the steady-state motion cannot be stabilized asymptotically by linear forces, since condition (2.8) becomes necessary in this case. However, the presence of internal dissipation forces $(a \neq 0)$ makes it possible for the motion to be asymptotically stabilized.

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# PERIODIC MOTIONS OF GYROSCOPIC SYSTEMS*

A.A. VORONIN and V.V. SAZONOV

A generalized conservative gyroscopic system is considered. It is shown that there is a two-parameter family of periodic solutions of the complete equations of motion of the system, close to the similar family of solutions of the precession equations.

1. Consider a conservative mechanical system which contains $l$ gyroscopes. We assume that the system position is defined by $2m+l$ generalized coordinates $x_1, \ldots, x_{2m}, \varphi_1, \ldots, \varphi_l$, where $\varphi_1, \ldots, \varphi_l$ are the angles of proper rotation of the gyroscopes, while $x=(x_1, \ldots, x_{2m})^T$ are parameters which characterize the directions of the gyroscope axes and the positions of the suspensions. We also assume that the system is described by the Lagrange function /1/

$$L=\frac{1}{2}\sum_{i,\,j=1}^{2m} a_{ij}(x)\,x_i^{\cdot}x_j^{\cdot}+\frac{1}{2}\sum_{k=1}^{l} C_k\Big(\varphi_k^{\cdot}+\sum_{i=1}^{2m} a_i^{(k)}(x)\,x_i^{\cdot}\Big)^2-\Pi(x)$$

Here, the dot denotes differentiation with respect to time $t$, $C_k$ are constants, and the symmetric matrix $A(x)=(a_{ij}(x))_{i,j=1}^{2m}$ is positive definite. The angles $\varphi_k$ are cyclical coordinates, and the corresponding first integrals are

$$\frac{\partial L}{\partial \varphi_k^{\cdot}}=C_k\Big(\varphi_k^{\cdot}+\sum_{i=1}^{2m} a_i^{(k)}(x)\,x_i^{\cdot}\Big)=h_k \quad (k=1, \ldots, l)$$

Using Rouse's method and introducing the notation

$$hg_{ij}(x)=\sum_{k=1}^{l} h_k\left(\frac{\partial a_i^{(k)}}{\partial x_j}-\frac{\partial a_j^{(k)}}{\partial x_i}\right) \quad (i, j=1, \ldots, 2m)$$

the equations of motion of the system can be written as

$$\sum_{j=1}^{2m} a_{ij}x_j^{\cdot\cdot}+\sum_{j,\,k=1}^{2m}\left(\frac{\partial a_{ij}}{\partial x_k}-\frac{1}{2}\frac{\partial a_{jk}}{\partial x_i}\right)x_j^{\cdot}x_k^{\cdot}=-h\sum_{j=1}^{2m} g_{ij}x_j^{\cdot}-\frac{\partial \Pi}{\partial x_i} \quad (i=1, \ldots, 2m) \tag{1.1}$$

These equations have the generalized energy integral

$$\frac{1}{2}\sum_{i,\,j=1}^{2m} a_{ij}(x)\,x_i^{\cdot}x_j^{\cdot}+\Pi(x)=\text{const} \tag{1.2}$$

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